A complexity dichotomy for graph orientation problems

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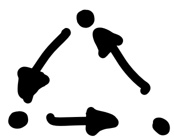
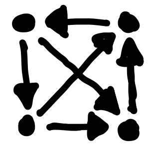
The graph orientation problem

Fix $\mathcal{F}$ finite set of finite directed graphs

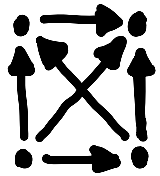
GOP($\mathcal{F}$) the "$\mathcal{F}$-free graph orientation problem":

INPUT: finite graph G

OUTPUT: YES/NO, depending on G admitting an orientation which does not embed any member of $\mathcal{F}$.

EXAMPLES ① $\mathcal{F} = \{$  ,  $\}$ OUTPUT: always YES:

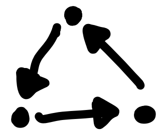
Acyclic orientation always possible! solvable in constant time

② $\mathcal{F} = \{$  ,  $\}$ OUTPUT: YES iff input graph does not contain a 4-clique. tractable

C_3 T_4

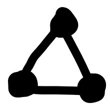
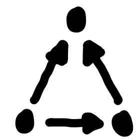
C_3 or T_4 embeds into every orient. of K_4. O/w acyclic orientation possible!

The graph orientation problem

EXAMPLES ③ $\mathcal{F} = \{ \bullet \to \bullet \to \bullet \to ,$  $\}$

follows from Gallai-Hasse-Roy-Vitaver Thm

OUTPUT : YES iff input graph is 3-colorable. NP-complete

$h : G \to$  $\rightsquigarrow$ orientation G' s.t. $h : G' \to$ 

$\mathcal{F}$-free orientation G'.

THEOREM (Bodirsky, Guzmán-Pro 2023)

If $\mathcal{F}$ consists of tournaments only, then $\mathrm{GOP}(\mathcal{F})$ is either tractable or NP-complete.

Connection to CSPs

Recall : $\mathbb{A}$ rel. structure with signature τ

- $CSP(\mathbb{A}) = \{\mathbb{B} \text{ finite } \tau\text{-structure} : \mathbb{B} \to \mathbb{A}\}$
- $CSP(\mathbb{A})$ is also the decision problem: $\mathbb{B} \in CSP(\mathbb{A})$?

$\boxed{\text{EXAMPLES}}$ ① $CSP(K_3) =$ "The 3-coloring problem"

② $CSP(\mathbb{Q}, <) =$ "decide if input is oriented acyclic graph"

③ $CSP(\mathbb{Z}, \{0\}, \{1\}, \underbrace{+, \cdot}_{\text{ternary relations}}) =$ "decide if Diophantine equation has integer solution."

Connection to CSPs

Bodirsky, Guzmán-Pro : $\mathcal{F}$ finite set of tournaments, exists graph $H_{\mathcal{F}}$ s.t. $CSP(H_{\mathcal{F}}) = GOP(\mathcal{F})$.

$H_{\mathcal{F}}$ is graph reduct of $D_{\mathcal{F}}$, the Fraïssé-Limit of $\mathcal{C}_{\mathcal{F}}$, the class of finite directed $\mathcal{F}$-free graphs.

$h : G \to H_{\mathcal{F}}$

pull back orientation $D_{\mathcal{F}}$ of $H_{\mathcal{F}}$ along h

$G' \hookrightarrow D_{\mathcal{F}}$,

so $G \hookrightarrow H_{\mathcal{F}}$

$\mathcal{F}$-free orientation G' of G

Connection to CSPs

$\mathcal{C}_{\mathcal{F}}$ the class of finite directed $\mathcal{F}$-free graphs is **finitely bounded**:

$\exists k \in \mathbb{N}$: $\mathbb{B} \in \mathcal{C}_{\mathcal{F}} \Leftrightarrow$ all substructures of $\mathbb{B}$ of size $\leq k$ belong to $\mathcal{C}_{\mathcal{F}}$.

take $k = \max\{2\} \cup \{|T| : T \in \mathcal{F}\}$

Conjecture (Bodirsky, Pinsker 2011)

If $\mathbb{A}$ is f.o. reduct of a **finitely bounded** homogeneous structure, CSP($\mathbb{A}$) is either tractable or NP-complete.

- $\mathbb{H}_{\mathcal{F}}$ is in scope of conjecture
- current proof doesn't use recent theory developed for that scope

↰ Smooth Approximations (Mottet, Pinsker 2020)

GOAL: Use recently developed theory to redo proof of complexity dichotomy of $GOP(\mathcal{F}) = CSP(H_{\mathcal{F}})$ to:

- test limitations of the theory
- aligning proof with previous applications of the theory to provide structured overview of current situation
- make proof amenable to further generalizations:
  - → $\mathcal{F}$ not necessarily consisting of tournaments
  - → considering edge coloring problems (captured by GMSNP) instead of edge orientation problems

- provide good challenge for beginning PhD student

Polymorphisms the symmetries of a CSP

- comp. complexity of CSP(A) captured by polymorphism clone $Pol(\mathbb{A}) := \bigcup_{n \in \mathbb{N}} \{h : \mathbb{A}^n \to \mathbb{A} \mid h \text{ homom.}\}$
- If $\mathbb{B}$ τ-structure, $h_1, \dots, h_n : \mathbb{B} \to \mathbb{A}$ homom., $f \in Pol(\mathbb{A})$ n-ary, then $f \circ (h_1, \dots, h_n)$ is homom. $\mathbb{B} \to \mathbb{A}$

"Polymorphisms are symmetries of solution space $\{h : \mathbb{B} \to \mathbb{A} \mid h \text{ homom.}\}$ of an instance $\mathbb{B}$ of CSP(A)"

few symmetries (only trivial ones) $\Rightarrow$ hard CSP

non-trivial symmetries $\Rightarrow$ CSP is tractable

Polymorphisms the symmetries of a CSP

THEOREM (Barto, Opršal, Pinsker 2017)

If $\mathbb{A}$ ω-cat. with only trivial symmetries (exists $\mathrm{Pol}(\mathbb{A}) \xrightarrow{u.c.} \mathrm{Proj}$) then $\mathrm{CSP}(\mathbb{A})$ is NP-complete.

THEOREM (Bulatov, Zhuk 2017)

If $\mathbb{A}$ finite, then $\mathrm{CSP}(\mathbb{A})$ is tractable iff exists $f \in \mathrm{Pol}(\mathbb{A})$ that is cyclic, i.e. $\forall x_1, \dots, x_n$ $f(x_1, \dots, x_n) = f(x_n, x_1, \dots, x_{n-1})$.

THEOREM (F., Pinsker)

$\mathcal{F}$ finite set of tournaments. Exactly one of the two holds:

(1) $H_{\mathcal{F}}$ has only trivial symmetries (exists $\mathrm{Pol}(H_{\mathcal{F}}) \xrightarrow{\text{u.c.}} \mathrm{Proj}$), so $\mathrm{CSP}(H_{\mathcal{F}}) = \mathrm{GOP}(\mathcal{F})$ is NP-complete.

(2) $\mathrm{Pol}(H_{\mathcal{F}})$ contains a canonical ternary function which is cyclic. In this case $\mathrm{CSP}(H_{\mathcal{F}})$ is tractable.

<u>pf. sketch</u>. If in case (2): Reduce $\mathrm{CSP}(H_{\mathcal{F}})$ to $\mathrm{CSP}(\mathbb{A})$, where $\mathbb{A}$ is finite orbit structure.

Problem: $\mathrm{CSP}(\mathbb{A})$ in general harder than $\mathrm{CSP}(H_{\mathcal{F}})$.

But: $\mathrm{Pol}(H_{\mathcal{F}})$ contains canonical ternary function, inducing cyclic polymorphism of $\mathbb{A}$.

Bulatov, Zhuk $\Rightarrow$ $\mathrm{CSP}(\mathbb{A})$ tractable, in particular $\mathrm{CSP}(H_{\mathcal{F}})$.

THEOREM (F., Pinsker)

$\mathcal{F}$ finite set of tournaments. Exactly one of the two holds:

(1) $H_\mathcal{F}$ has only trivial symmetries (exists $\mathrm{Pol}(H_\mathcal{F}) \xrightarrow{u.c.} \mathrm{Proj}$), so $\mathrm{CSP}(H_\mathcal{F}) = \mathrm{GOP}(\mathcal{F})$ is NP-complete.

(2) $\mathrm{Pol}(H_\mathcal{F})$ contains a canonical ternary function which is cyclic. In this case $\mathrm{CSP}(H_\mathcal{F})$ is tractable.

<u>pf. sketch</u>. If not in case (2): The canonical polymorphisms $\mathrm{Pol}(H_\mathcal{F})^{can} \subseteq \mathrm{Pol}(H_\mathcal{F})$ consist of trivial symmetries only.

Exist $\phi : \mathrm{Pol}(H_\mathcal{F})^{can} \xrightarrow{u.c.} \mathrm{Proj}$.

Smooth Approx.: $\exists$ mapping $\psi : \mathrm{Pol}(H_\mathcal{F}) \to \mathrm{Pol}(H_\mathcal{F})^{can}$ s.t.

$\phi \circ \psi : \mathrm{Pol}(H_\mathcal{F}) \xrightarrow{u.c.} \mathrm{Proj}$. Barto, Opršal, Pinsker $\Rightarrow$ $\mathrm{CSP}(H_\mathcal{F})$ NP-c.

Thank you!

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