# DECIDABILITY OF INTERPRETABILITY

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European Research Council
Established by the European Commission

SyG POCOCOP
GA 101071674

LICS '26 JULY 2026

# CSPs

$\mathbb{A} = (A; R_1, \dots, R_n)$ relational structure

| CSP($\mathbb{A}$) |
| --- |
| INPUT: finite rel. structure $\mathbb{I}$ |
| QUESTION: $\mathbb{I} \xrightarrow{\text{hom.}} \mathbb{A}$ ? |

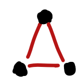
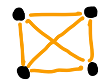
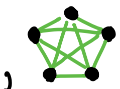
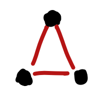
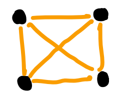
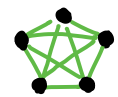
**Examples**
- CSP($K_2$) = "graph bipartite?"
- CSP($K_3$) = "graph 3-colorable?"
- CSP($\mathbb{Z}, +, \times, 0, 1$) = "solving Dioph. equations"

- every computational problem is P-time equiv. to CSP

(Bodirsky + Grohe 2008)

- P vs. NP-c. dichotomy for finite-domain CSPs + nice hardness criterion

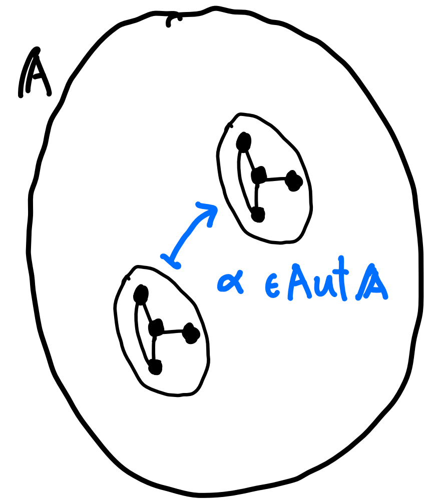
(Bulatov, Zhuk 2017)

| Edge-coloring problem , ,  |
| --- |
| INPUT: finite graph |
| QUESTION: Can edges be colored with , , without creating: , ,  ? |

**Conjecture:** P vs. NP-c. dichotomy for fo-reducts of fbh structures + nice hardness criterion.

(Bodirsky + Pinsker 2011)

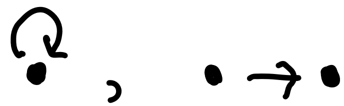
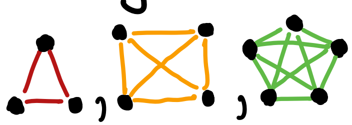
# Finitely bounded homogeneous (fbh)




A structure $\mathbb{A}$ is:

- **homogeneous** if isomorphism between finite substructures extend to automorphisms of $\mathbb{A}$
- **finitely bounded** if exist structures $\mathbb{F}_1, \dots, \mathbb{F}_n$ (the bounds) $\forall$ finite $\mathbb{C}$ : $\mathbb{C}$ embeds into $\mathbb{A}$ $\Leftrightarrow$ no $\mathbb{F}_i$ embeds into $\mathbb{C}$

**Examples**

- $(\mathbb{Q}, <)$ — bounds: 
- Random graph — bounds: 
- Flim{ edge-colored finite  -free graphs }

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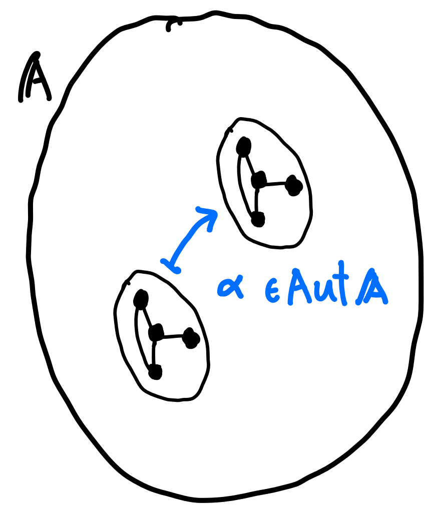


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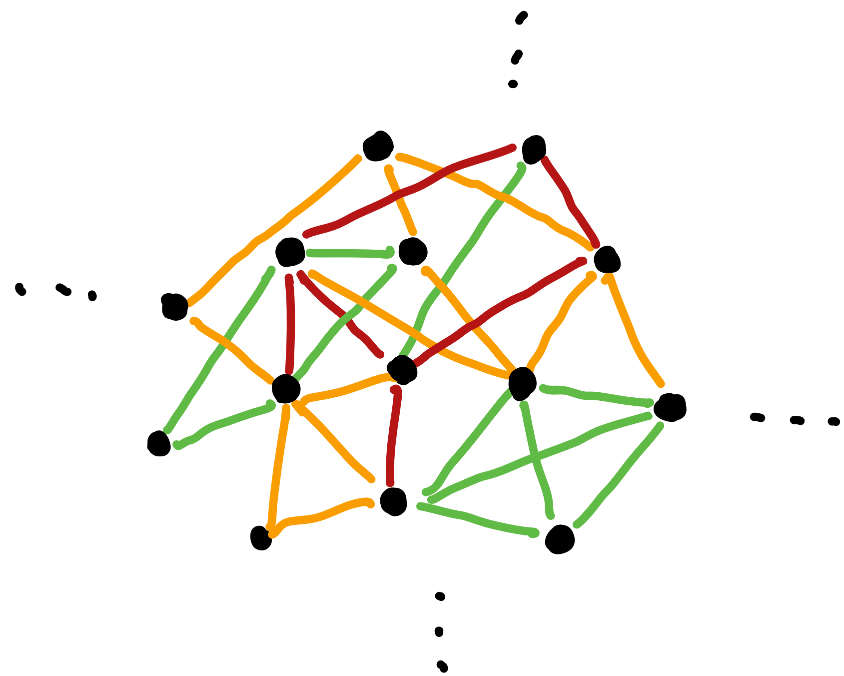
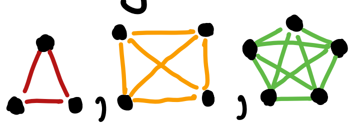


- Flim{ edge-colored finite  -free graphs }

# (Some) tools for the conjecture

## Polymorphisms

$$\mathrm{Pol}\,\mathbb{A} = \bigcup_{n\geq\omega} \{\mathbb{A}^n \xrightarrow{\text{hom.}} \mathbb{A}\}$$

**$\mathbb{A}$ $\omega$-cat:**

- $\mathrm{Pol}\,\mathbb{A} = \mathrm{Pol}\,\mathbb{B} \Rightarrow$

  $\mathrm{CSP}\,\mathbb{A} \sim_{\text{P-time}} \mathrm{CSP}\,\mathbb{B}$

  (Bodirsky + Nešetřil)

- even for $\mathrm{Pol}\,\mathbb{A} \cong_{\text{top}} \mathrm{Pol}\,\mathbb{B}$

  (Bodirsky + Pinsker)

**Question 1:** How hard to check if $\mathrm{Pol}\,\mathbb{A} = \mathrm{Pol}\,\mathbb{B}$ or $\mathrm{Pol}\,\mathbb{A} \cong_{\text{top}} \mathrm{Pol}\,\mathbb{B}$ ?

## Model-complete cores

$\mathbb{A}$ is model-complete core if $\forall e \in \mathrm{End}\,\mathbb{A}$ $e|_F \in \mathrm{Aut}\,\mathbb{A}|_F$ $\forall F \subseteq A$ finite.

- $\mathbb{A}$ $\omega$-cat. exist unique m-c core $\mathbb{A}^c$ that is homom. equiv. to $\mathbb{A}$

  (Bodirsky 2005)

**Examples**

- $(\mathbb{Q};<)^c = (\mathbb{Q};<)$
- $(\mathbb{Q};\leq)^c = (\{0\};\leq)$
- Random graph $^c = K_\omega$
- $K_{\omega,\omega}{}^c = K_2$

**Question 2:** How hard is it to find the model-complete core of $\mathbb{A}$ ?

# Question 1A: "How hard to check if $\mathrm{Pol}\,\mathbb{A} = \mathrm{Pol}\,\mathbb{B}$?"

**Thm.** (Bodirsky + Pinsker + Tsankov 2011) "Decidability of Definability"

If $\mathbb{A}, \mathbb{B} \leq_{fo} \mathbb{C}$, $\mathbb{C}$ fbh + Ramsey, then can decide if

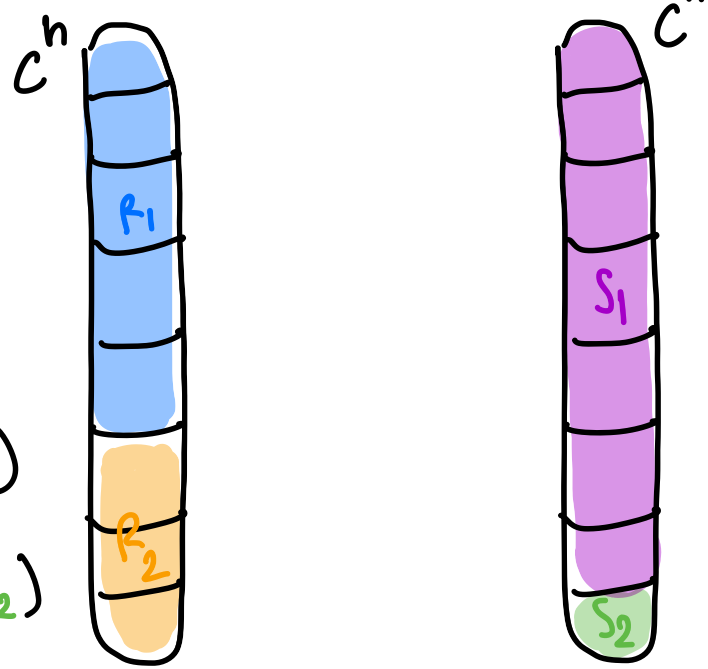
$$\mathrm{Pol}\,\mathbb{A} = \mathrm{Pol}\,\mathbb{B}.$$

sketch of proof. $\mathrm{Pol}\,\mathbb{A} \neq \mathrm{Pol}\,\mathbb{B} \iff$ wlog $\exists f \in \mathrm{Pol}\,\mathbb{A} \setminus \mathrm{Pol}\,\mathbb{B}$

$\iff \exists f \in \mathrm{Pol}\,\mathbb{A} \setminus \mathrm{Pol}\,\mathbb{B}$ acting on $(\mathrm{Aut}\,\mathbb{C} \curvearrowright C^n)$-orbits $\forall n$.

$\mathbb{A} = (C; R_1, R_2)$

$\mathbb{B} = (C; S_1, S_2)$



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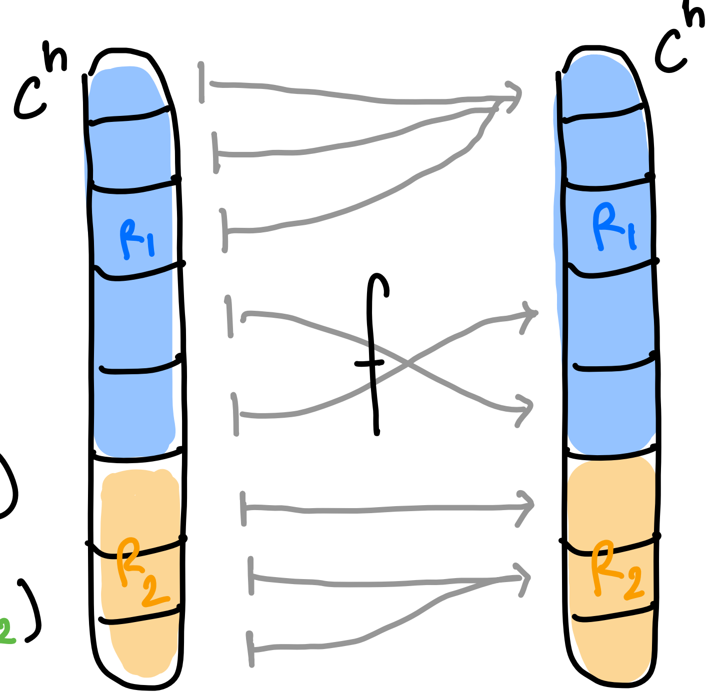

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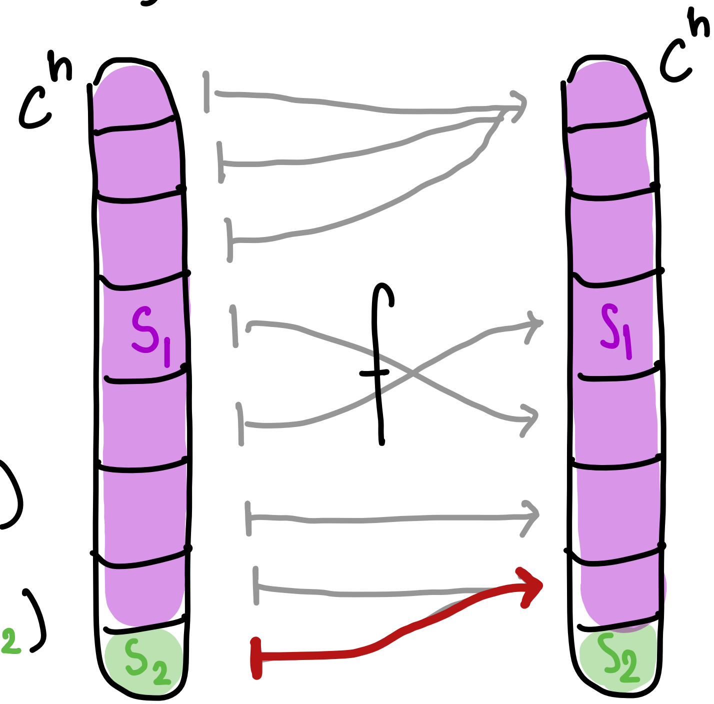

# Question 1A: "How hard to check if $\mathrm{Pol}\,\mathbb{A} = \mathrm{Pol}\,\mathbb{B}$?"

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$\mathbb{A}, \mathbb{B}$ are pp-interdefinable

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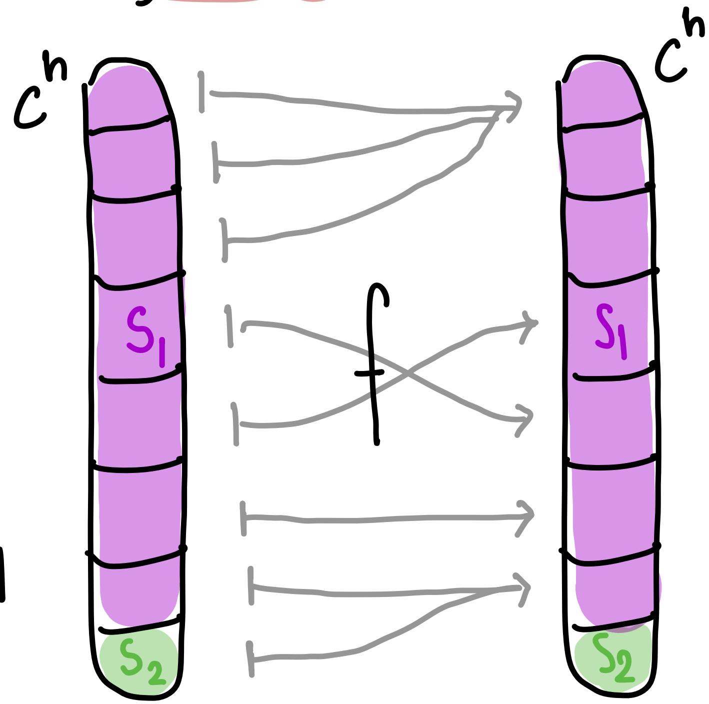


fbh: for large enough $n$ (computable) can decide if operation on $(\mathrm{Aut}\,\mathbb{C} \curvearrowright C^n)$-orbits is induced by polymorphism of $\mathbb{A}$ or $\mathbb{B}$. $\square$

# Question 2: "How hard is it to find m-c core?"

**Thm.** (F. + Pinsker 2026)
If $\mathbb{A} \leq_{fo} \mathbb{C}$, $\mathbb{C}$ fbh + Ramsey, then can compute the m-c core $\mathbb{A}^c$ of $\mathbb{A}$.

Sketch of proof.

**$\mathbb{C}$ finite:** Choose $f \in \mathrm{End}\,\mathbb{A}$ with $f(A)$ minimal.

- $f(\mathbb{A}) \subseteq \mathbb{A}$ homom. equiv. to $\mathbb{A}$
- $\mathrm{Aut}\, f(\mathbb{A}) = \mathrm{End}\, f(\mathbb{A})$

**$\mathbb{C}$ general:** Choose $f \in \mathrm{End}\,\mathbb{A}$ acting on $(\mathrm{Aut}\,\mathbb{C} \curvearrowright C^n)$-orbits $\forall n$ s.t. $f(A)$ intersects minimal set of orbits.

- $\mathbb{A}^c =$ "$f(\mathbb{A})$" $\square$

# Question 1B: "How hard to check if $\mathrm{Pol}\,\mathbb{A} \cong_{top} \mathrm{Pol}\,\mathbb{B}$?"

**Thm.** (F. + Pinsker 2026) "Decidability of Interpretability"

If $\mathbb{A} \leq_{fo} \mathbb{C}$, $\mathbb{B} \leq_{fo} \mathbb{D}$ and $\mathbb{C}, \mathbb{D}$ fbh + Ramsey, $\mathbb{A}^c, \mathbb{B}^c$ without algebraicity, 1-transitive, then can decide if

$$\mathrm{Pol}\,\mathbb{A}^c \cong_{top} \mathrm{Pol}\,\mathbb{B}^c$$

$\mathbb{A}^c, \mathbb{B}^c$ are pp-bi-interpretable

**Main ingredients:**

- m-c core is computable
- (Rubin 1994) $\mathbb{A}, \mathbb{B}$ $\omega$-cat. without algebraicity:

$$\mathrm{Aut}\,\mathbb{A} \cong_{top} \mathrm{Aut}\,\mathbb{B} \Leftrightarrow \exists \text{ bijection } \phi: A \to B$$
$$\phi \,\mathrm{Aut}\,\mathbb{A}\, \phi^{-1} = \mathrm{Aut}\,\mathbb{B}$$

- with additional assumption the same holds for $\mathrm{Pol}\,\mathbb{A}^c$, $\mathrm{Pol}\,\mathbb{B}^c$.

# Beyond decidability: measuring complexity

- First-order reducts of fbh. structures : can be finitely represented

makes sense to ask: "Is $\mathrm{Pol}\,\mathbb{A} \cong_{\mathrm{Top}} \mathrm{Pol}\,\mathbb{B}$ decidable?"

- $\omega$-cat. structures: not fin. representable $\rightsquigarrow$ decidability makes no sense

But: can be encoded as 0-1-seq.
Their collection forms Borel space $\cong \mathbb{R}$.

Equiv. rel. $\sim$ on Borel space $X$ is smooth, if exists $I: X \xrightarrow{\text{Borel}} \mathbb{R}$ s.t. $x \sim y \Leftrightarrow I(x) = I(y)$

# Smoothness of top. isomorphism

**Thm.** (Nies + Paolini 2024)

The equivalence rel.

$$A \sim B \;:\Leftrightarrow\; \operatorname{Aut} A \cong_{top} \operatorname{Aut} B$$

is smooth for $\omega$-cat. structures without algebraicity.

**Key realization**: $\operatorname{Aut} A \cong_{top} \operatorname{Aut} B \iff \exists$ bij. $\phi : A \to B$ s.t.
$\phi \operatorname{Aut} A\, \phi^{-1} = \operatorname{Aut} B$.

**Thm.** (F. + Pinsker 2026) $A \sim B :\Leftrightarrow \operatorname{Pol} A \cong_{top} \operatorname{Pol} B$ is smooth for $\omega$-cat., 1-transitive model-complete cores without algebraicity.

Thank you!