DECIDABILITY OF INTERPRETABILITY

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CSPs

$\mathbb{A} = (A; R_1, \ldots, R_n)$ relational structure

CSP($\mathbb{A}$)
INPUT: finite rel. structure $\mathbb{I}$
QUESTION: $\mathbb{I} \xrightarrow{\text{hom.}} \mathbb{A}$?

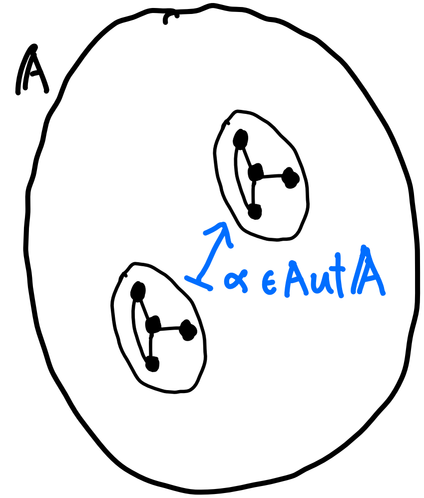
Examples

- CSP(K_2) = "graph bipartite?"
- CSP(K_3) = "graph 3-colourable ?"
- CSP($\mathbb{Z}, +, \times, 0, 1$) = "solving Dioph. equations"

- every computational problem is P-time equiv. to CSP (Bodirsky + Grohe 2008)
- P vs. NP-c. dichotomy for finite-domain CSPs + nice hardness criterion. (Bulatov, Zhuk 2017)
- **Conjecture:** P vs. NP-c. dichotomy for first-order reducts of fbh structures + nice hardness criterion. (Bodirsky + Pinsker 2011)

Finitely bounded homogeneous (fbh)

$\mathbb{A}$ is ...



- **homogeneous** if isomorphism between finite substructures extend to automorphisms of $\mathbb{A}$
- **finitely bounded** if exists $n \in \omega$ s.t.

  $\forall$ finite $\mathbb{C}$:

$$\mathbb{C} \hookrightarrow \mathbb{A} \iff \text{all size } n \text{ substructures of } \mathbb{C} \text{ embed into } \mathbb{A}$$

Examples
- finite structures
- $(\mathbb{Q}, <)$
- Random graph
- Random ordered graph
- Flim $\left\{ \begin{array}{l} \text{finite red/blue-edge-coloured} \\ \text{graphs w/o monochromatic triangles} \end{array} \right\}$

(Some) tools for the conjecture

Polymorphisms

$$\mathrm{Pol}\,\mathbb{A} = \bigcup_{n<\omega} \{\mathbb{A}^n \xrightarrow{\text{hom.}} \mathbb{A}\}$$

$\mathbb{A}$ ω-cat:

- $\mathrm{Pol}\,\mathbb{A} = \mathrm{Pol}\,\mathbb{B} \Rightarrow$

  $\mathrm{CSP}\,\mathbb{A} \sim_{\text{P-time}} \mathrm{CSP}\,\mathbb{B}$

  (Bodirsky + Nešetřil)

- even for $\mathrm{Pol}\,\mathbb{A} \cong_{top} \mathrm{Pol}\,\mathbb{B}$

  (Bodirsky + Pinsker)

Question 1: How hard to check if $\mathrm{Pol}\,\mathbb{A} = \mathrm{Pol}\,\mathbb{B}$ or $\mathrm{Pol}\,\mathbb{A} \cong_{top} \mathrm{Pol}\,\mathbb{B}$?

Model-complete cores

$\mathbb{A}$ is model-complete core if $\overline{\mathrm{Aut}\,\mathbb{A}} = \mathrm{End}\,\mathbb{A}$.

endomorphisms look like automorphisms on finite sets

- $\mathbb{A}$ ω-cat. exist unique m-c core $\mathbb{A}^c$ that is hom. equiv. to $\mathbb{A}$

  (Bodirsky 2005)

Examples

- $(\mathbb{Q};<)^c = (\mathbb{Q};<)$
- $(\mathbb{Q};\leq)^c = (\{0\};\leq)$
- Random graph$^c = K_\omega$
- $K_{\omega,\omega}{}^c = K_2$

Question 2: How hard is it to find the model-complete core of $\mathbb{A}$?

Question 1A: "How hard to check if $\mathrm{Pol}\,\mathbb{A} = \mathrm{Pol}\,\mathbb{B}$?"

Thm. (Bodirsky + Pinsker + Tsankov 2011) "Decidability of Definability"

If $\mathbb{A}, \mathbb{B} \leq_{fo} \mathbb{C}$, $\mathbb{C}$ fbh + Ramsey, then can decide if

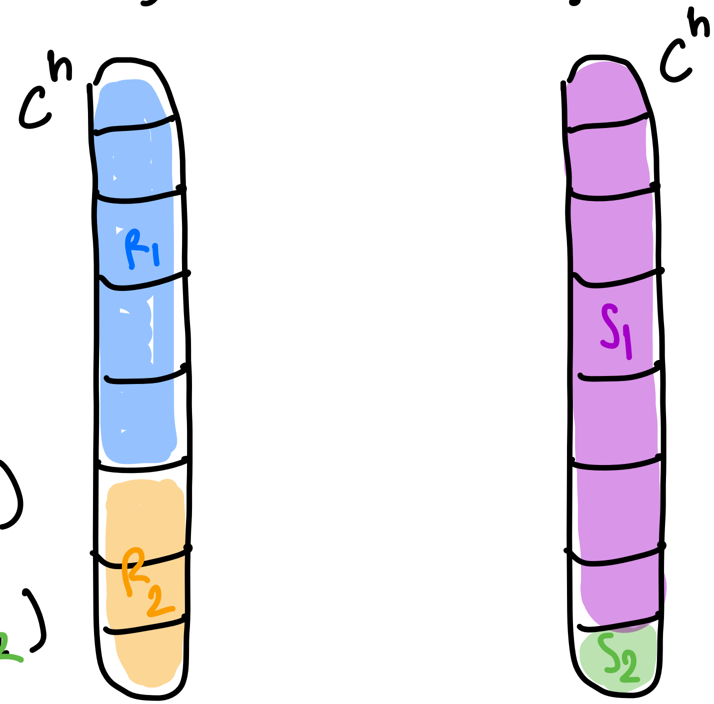
$$\mathrm{Pol}\,\mathbb{A} = \mathrm{Pol}\,\mathbb{B}.$$

sketch of proof. $\mathrm{Pol}\,\mathbb{A} \neq \mathrm{Pol}\,\mathbb{B} \Leftrightarrow$ wlog $\exists f \in \mathrm{Pol}\,\mathbb{A} \setminus \mathrm{Pol}\,\mathbb{B}$

$\Leftrightarrow \exists f \in \mathrm{Pol}\,\mathbb{A} \setminus \mathrm{Pol}\,\mathbb{B}$ acting on $(\mathrm{Aut}\,\mathbb{C} \curvearrowright C^n)$-orbits $\forall n$.

$\mathbb{A} = (C; R_1, R_2)$

$\mathbb{B} = (C; S_1, S_2)$



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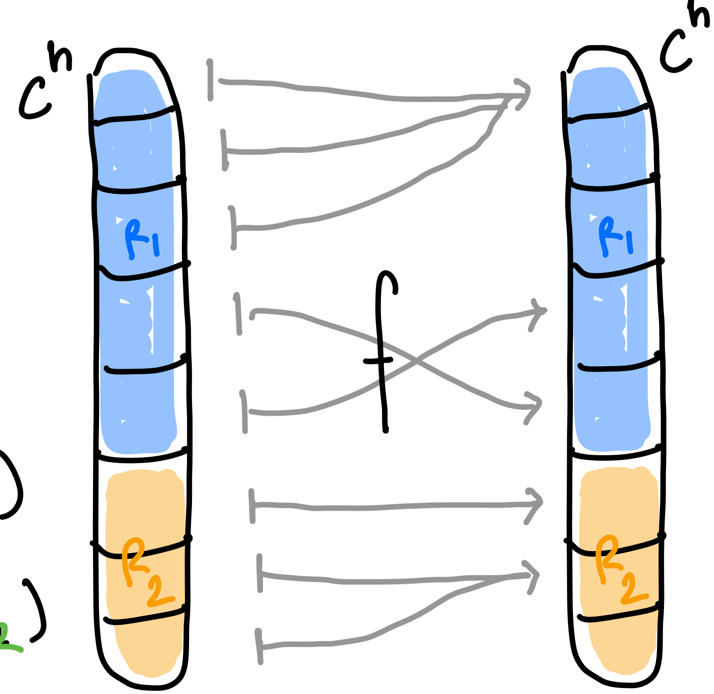

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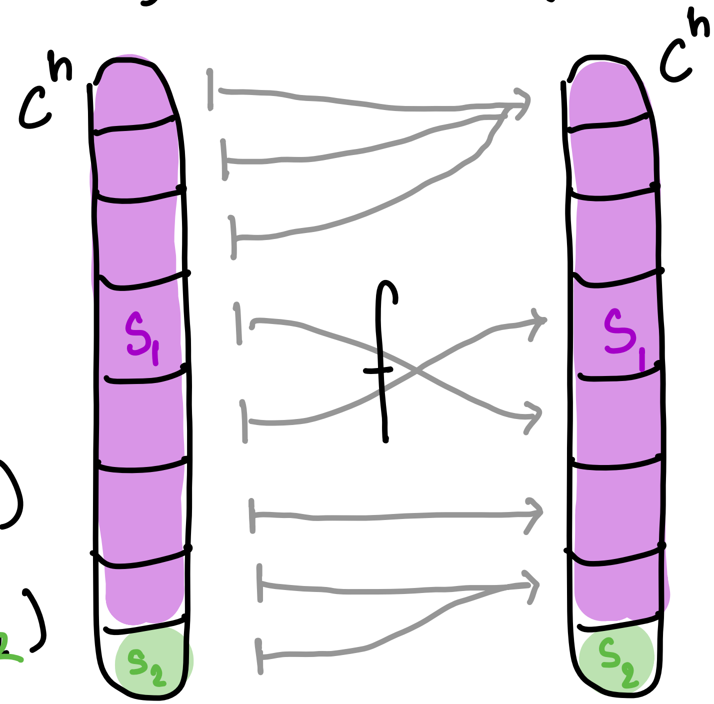

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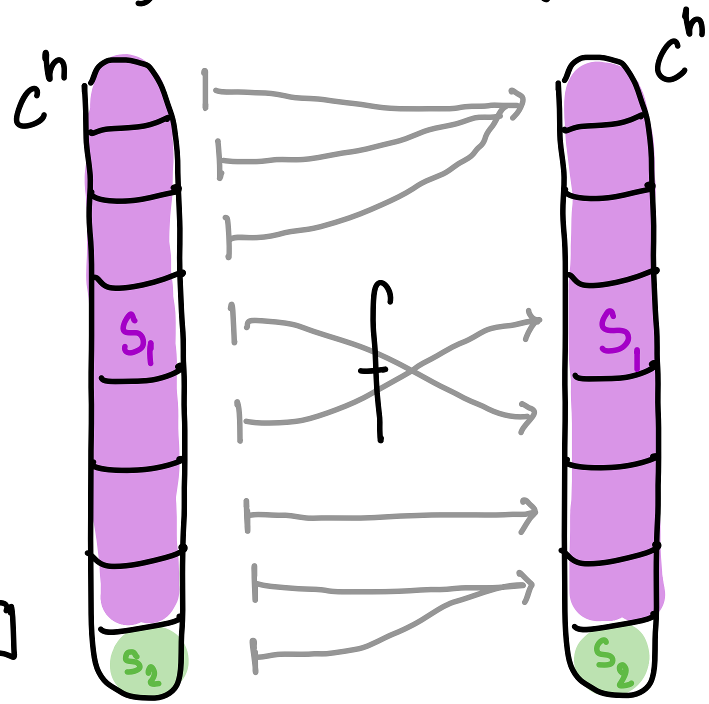
fbh: for large enough n (computable) can decide if operation on $(\mathrm{Aut}\,\mathbb{C} \curvearrowright C^n)$-orbits is induced by polymorphism of $\mathbb{A}$ or $\mathbb{B}$.



$\square$

Question 2: "How hard is it to find m-c core?" 5

Thm. (F. + Pinsker 2026)
If $\mathbb{A} \leq_{f.o.} \mathbb{C}$, $\mathbb{C}$ fbh + Ramsey, then can compute the m-c core $\mathbb{A}^c$ of $\mathbb{A}$.

Sketch of proof. Recall $\overline{\mathrm{Aut}\,\mathbb{A}^c} = \mathrm{End}\,\mathbb{A}$.

$\mathbb{C}$ finite: Choose $f \in \mathrm{End}\,\mathbb{A}$ with $f(A)$ minimal.

- $f(\mathbb{A}) \subseteq \mathbb{A}$ homomorphically equiv. to $\mathbb{A}$
- $\mathrm{Aut}\, f(\mathbb{A}) = \mathrm{End}\, f(\mathbb{A})$: for $g \in \mathrm{End}\, f(\mathbb{A})$, f and $g \circ f$ have same image $\Rightarrow g$ bijective $\Rightarrow g \in \mathrm{Aut}\, f(\mathbb{A})$.

$\mathbb{C}$ general: Choose $f \in \mathrm{End}\,\mathbb{A}$ acting on $(\mathrm{Aut}\,\mathbb{C} \curvearrowright C^n)$-orbits $\forall n$ s.t. $f(A)$ intersects minimal set of $\mathrm{Aut}\,\mathbb{C}$-orbits.

- $\mathbb{A}^c =$ "$f(\mathbb{A})$"

$\square$

Question 1B: "How hard to check if $\mathrm{Pol}\,\mathbb{A} \cong_{top} \mathrm{Pol}\,\mathbb{B}$?" 6

Thm. (F. + Pinsker 2026) "Decidability of Interpretability"

If $\mathbb{A} \leq_{fo} \mathbb{C}$, $\mathbb{B} \leq_{fo} \mathbb{D}$, $\mathbb{C}, \mathbb{D}$ fbh + Ramsey, $\mathbb{A}^c, \mathbb{B}^c$ without algebraicity, transitive, then can decide if:

$$\mathrm{Pol}\,\mathbb{A}^c \cong_{top} \mathrm{Pol}\,\mathbb{B}^c.$$

$\mathbb{A}^c, \mathbb{B}^c$ are pp-bi-interpretable

- in the scope of the infinite-domain CSP dichotomy conjecture wlog the case (see: Christoph Spiess)

- annoying; although often the case in practice.

Main ingredients: • computability of model-complete core

• (Rubin 1994) $\mathbb{A}, \mathbb{B}$ ω-cat. without algebraicity, then:

$$\mathrm{Aut}\,\mathbb{A} \cong_{top} \mathrm{Aut}\,\mathbb{B} \Rightarrow \begin{array}{l} \exists \text{ bij. } \phi: A \to B \\ \phi\, \mathrm{Aut}\,\mathbb{A}\, \phi^{-1} = \mathrm{Aut}\,\mathbb{B} \end{array}$$

• Same for Pol with transitive model-complete core assumption.

Measuring the complexity of equiv. rel.

- First-order reducts of fbh. structures : can be encoded finitely
- ω-cat. structures : can be encoded as 0-1-seq.
  $\{\omega\text{-cat. struct. on } \omega\} = 2^{\omega}$,
  standard Borel space.

Equiv. rel. $\sim$ on Borel space X is smooth, if exists $I : X \xrightarrow{\text{Borel}} \mathbb{R}$ s.t. $x \sim y \Leftrightarrow I(x) = I(y)$.

Smoothness of top. isomorphism

Thm. (Nies + Paolini 2024)
The equivalence rel.

$$A \sim \mathbb{B} \;:\Leftrightarrow\; \mathrm{Aut}\,\mathbb{A} \cong_{top} \mathrm{Aut}\,\mathbb{B}$$

is smooth for ω-cat. structures without algebraicity.

Key realization: $\mathrm{Aut}\,\mathbb{A} \cong_{top} \mathrm{Aut}\,\mathbb{B} \;\Leftrightarrow\; \exists$ bij. $\phi : \mathbb{A} \to \mathbb{B}$ s.t.

$$\phi\,\mathrm{Aut}\,\mathbb{A}\,\phi^{-1} = \mathrm{Aut}\,\mathbb{B}.$$

Cor. (F. + Pinsker 2026) $\mathbb{A} \sim \mathbb{B} :\Leftrightarrow \mathrm{Pol}\,\mathbb{A} \cong_{top} \mathrm{Pol}\,\mathbb{B}$ is smooth for ω-cat. transitive model-complete cores without algebraicity

Thank you!

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