DECIDABILITY OF INTERPRETABILITY

ROMAN FELLER (joint w. M. Pinsker)

TU WIEN

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SEMINAR ON FOUNDATIONS OF COMPUTING

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Background / Motivation (I)

$\mathbb{A} = (A; R_1, R_2, \ldots, R_k)$ relational structure

CSP $\mathbb{A}$ "Constraint Satisfaction Problem"
INPUT : finite structure $\mathbb{I}$ (same signature as $\mathbb{A}$)   DECIDE : $\exists$ homomorphism $\mathbb{I} \to \mathbb{A}$?

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4. CSP$(\mathbb{Z}; \underbrace{+, \times}_{\text{3-ary}}, \underbrace{0, 1}_{\text{unary}})$ = Diophantine eq. problem

Background / Motivation (II)

$\boxed{\text{Feder-Vardi Conjecture: If } A \text{ finite, } CSP(A) \text{ in P or NP-c.}}$

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Combine solutions:

$$\phi_1, \ldots, \phi_n : \mathbb{I} \to \mathbb{A} \quad \rightsquigarrow \quad f \circ (\phi_1, \ldots, \phi_n) : \mathbb{I} \to \mathbb{A}$$

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"More polymorphisms $\Rightarrow$ less unstructured search necessary"

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$\boxed{\textbf{Example}}$ ① $\mathrm{Pol}(K_3) = \{\alpha \circ \mathrm{proj} \mid \alpha \in \mathrm{Aut}(K_3)\}$ NP-c

② $\mathrm{Pol}(K_2) \ni \text{majority} : (K_2)^3 \to K_2$ P

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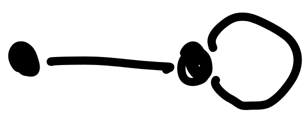
Example

① K_2 core

② K_3 core

③  no core

④ K_ω no core

⑤ $\cdots$ •—•—• $\cdots$ no core

⑥ $\cdots \to \bullet \to \bullet \to \bullet \to \cdots$ core

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Nice properties: ① $\mathrm{CSP}(\mathbb{C}) \sim_{\text{P-time}} \mathrm{CSP}(\mathbb{C}, \{c_1\}, \ldots, \{c_n\})$

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Observation. Every finite structure homom. equivalent to unique core structure.

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- CSP's form good framework to study comp. complexity; let's use it!

- tractability of some finite-domain PCSP's only witnessed by infinite domain CSP's

Mottet 2025 ; Pinsker, Rydval, Schöbi, Spiess 2025

Infinite domain CSP's

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Want class of structures with CSP's in NP.

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Recall: $\mathrm{CSP}(\mathbb{Z}, +, \times, \{0\}, \{1\})$ undecidable

Want class of structures with CSP's in NP.

Def. $\mathbb{A}$ is homogeneous if isomorphisms between finite substructures extend to automorphisms of $\mathbb{A}$.

$\mathbb{A}$ is finitely bounded if $\exists\ \mathcal{B} = \{\mathbb{B}_1, \ldots, \mathbb{B}_k\}$ s.t.

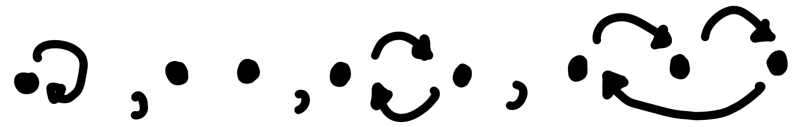
$\forall$ fin. $\mathbb{C}$ same sig. as $\mathbb{A}$:

$$\mathbb{C} \underset{\text{embedding}}{\hookrightarrow} \mathbb{A} \iff \mathbb{B}_1 \not\hookrightarrow \mathbb{C}, \ldots, \mathbb{B}_k \not\hookrightarrow \mathbb{C}$$

embedding = isomorphism onto substructure of codomain

Finitely bounded homogeneous structures

Examples

① $(\mathbb{Q}, <)$ $\quad \mathcal{B} = \{$  $\}$

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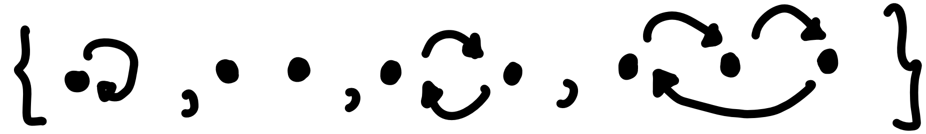
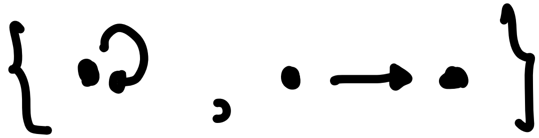
1. $(\mathbb{Q}, <)$ $\mathcal{B} =$ 

2. Rado Graph $\mathcal{B} =$ 

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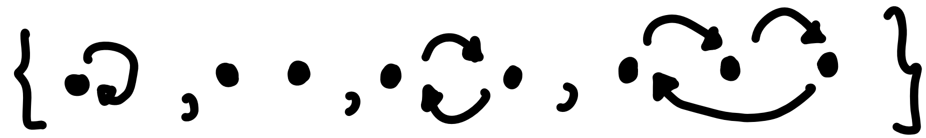
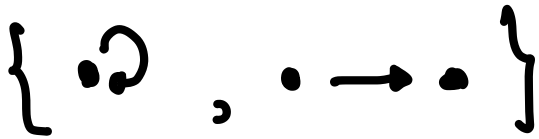
① $(\mathbb{Q}, <)$ $\qquad \mathcal{B} =$ 

② Rado Graph $\qquad \mathcal{B} =$ 

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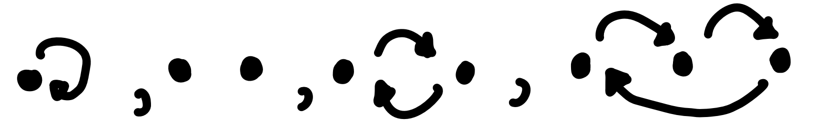
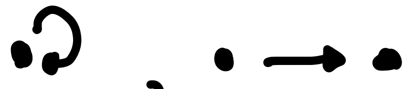
Examples ① $(\mathbb{Q}, <)$ $\mathcal{B} = \{$  $\}$

② Rado Graph $\mathcal{B} = \{$  $\}$

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Fraisse's Thm.

$\mathcal{C}$ class of fin. rel. structures s.t.

① $A \subseteq B$, $B \in \mathcal{C} \Rightarrow A \in \mathcal{C}$

② $\mathcal{C}$ has the amalgamation property

③ $\mathcal{C}$ countable up to isom.

$\Rightarrow \exists$ countable homogeneous $\mathrm{Flim}\,\mathcal{C}$ s.t. $\mathrm{age}(\mathrm{Flim}\,\mathcal{C}) = \mathcal{C}$

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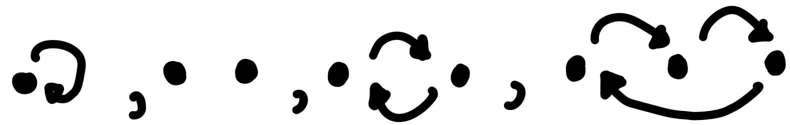
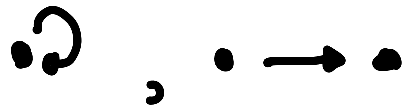
Examples ① $(\mathbb{Q}, <)$ $\mathcal{B} = \{$  $\}$

② Rado Graph $\mathcal{B} = \{$  $\}$

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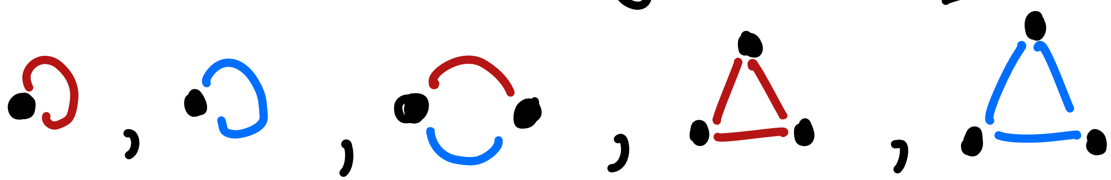
$\mathcal{B} = \{$  $\}$

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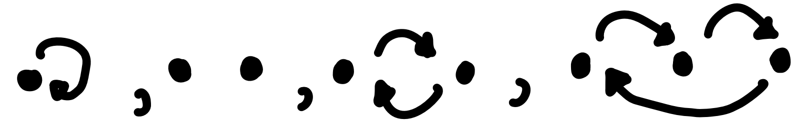
Examples ① $(\mathbb{Q}, <)$ $\quad \mathcal{B} = \{$  $\}$

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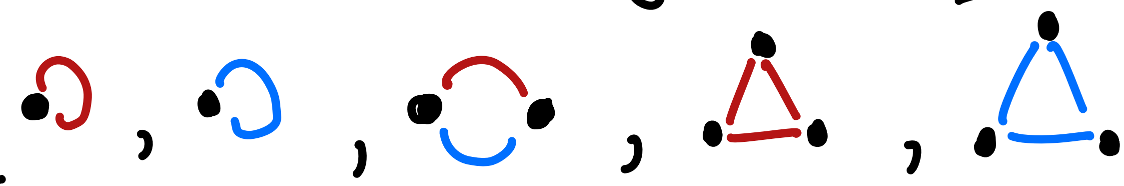
$\mathcal{B} = \{$  $\}$

Def. $\mathbb{A} \leq_{fo} \mathbb{B}$ if same domain + all rel. of $\mathbb{A}$ fo-definable in $\mathbb{B}$. ($\mathbb{B}$ homog. $\Rightarrow$ wlog quantifier free form.)

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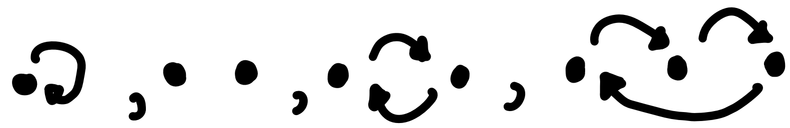
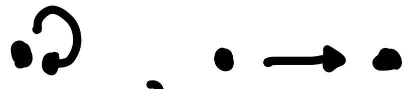
Examples ① $(\mathbb{Q}, <)$ $\quad \mathcal{B} = \{$  $\}$

② Rado Graph $\quad \mathcal{B} = \{$  $\}$

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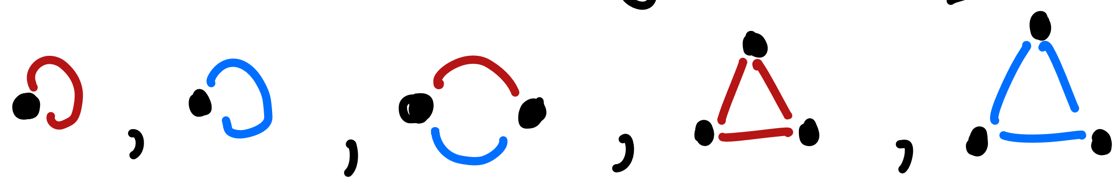
$\mathcal{B} = \{$  $\}$

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Why?

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- can verify if $\mathbb{I} \to \mathbb{A}$, $i_k \mapsto b_k$ is homomorphism (homogeneous $\Rightarrow$ rel's of $\mathbb{A}$ have quant. free def.)

Algebraic approach to infinite-domain CSP's

Thm. (Bodirsky + Nešetřil) $\mathbb{A}, \mathbb{B}$ ω-cat. (includes f.b.h.)

$\mathrm{Pol}\,\mathbb{A} = \mathrm{Pol}\,\mathbb{B} \Rightarrow \mathrm{CSP}\,\mathbb{A} \sim_{p\text{-time}} \mathrm{CSP}\,\mathbb{B}$

$\rightsquigarrow$ First regime of inf.-dom. CSP research.

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Thm. (Bodirsky + Pinsker + Tsankov) $\mathbb{A}, \mathbb{B} \leq_{fo} \mathbb{C}$, $\mathbb{C}$ f.b.h. + Ramsey.
Can decide if $\mathrm{Pol}\,\mathbb{A} = \mathrm{Pol}\,\mathbb{B}$

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⇝ Modern approach to inf. dom. research:

"Two struct. $\mathbb{A}, \mathbb{B}$ equiv. if $\mathrm{Pol}\,\mathbb{A}^{core} \cong_{top} \mathrm{Pol}\,\mathbb{B}^{core}$"

Thm. (F. + Pinsker) $\mathbb{A} \leq_{fo} \mathbb{C}$, $\mathbb{B} \leq_{fo} \mathbb{D}$, $\mathbb{C}, \mathbb{D}$ f.b.h, Ramsey, without algebraicity, transitive.
Can decide if $\mathrm{Pol}\,\mathbb{A}^{core} \cong_{top} \mathrm{Pol}\,\mathbb{B}^{core}$.

Thm. (F. + Pinsker) $\mathbb{A} \leq_{fo} \mathbb{C}$, $\mathbb{B} \leq_{fo} \mathbb{D}$, $\mathbb{C}, \mathbb{D}$ f.b.h, Ramsey, without algebraicity, transitive. Can decide if $\mathrm{Pol}\,\mathbb{A}^{core} \cong_{top} \mathrm{Pol}\,\mathbb{B}^{core}$.

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1. Show that m-c cores are computable.

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Computing the core

Def. $\mathbb{A}$ is model-complete core if $\forall e \in \mathrm{End}\,\mathbb{A}$ $\forall F \subseteq_{\text{fin.}} A$ $\exists \alpha \in \mathrm{Aut}\,\mathbb{A}$: $e|_F = \alpha|_F$.

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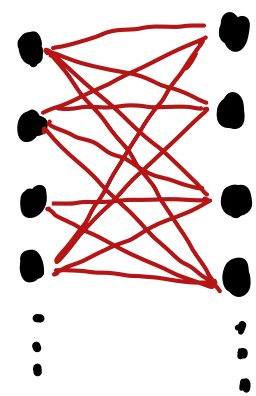
Examples ① $(\mathbb{Q},<)^{core} = (\mathbb{Q},<)$

② $(\text{Rado})^{core} = K_\omega$

③ $(K_{\omega,\omega})^{core} = K_2$

$K_{\omega,\omega}$:



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Example $f: \overbrace{(\mathbb{N},=)}^{\mathbb{A}} \to (\mathbb{N},=)$ is $(\mathbb{A},\mathbb{A})$ canonical iff injective or $|f(\mathbb{N})| = 1$.

Computing the core

Not all functions $f: \mathbb{A} \to \mathbb{B}$ are $(\mathbb{A},\mathbb{B})$-can.

If $\mathbb{A}$ Ramsey: can approximate f by can. f':

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$$f_{orb}(O) = \{O_1, \dots, O_n\} \text{ smallest s.t. } f(O) \subseteq O_1 \cup \dots \cup O_n$$

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⇝ $\mathbb{A}$ Ramsey $\Rightarrow$ wlog functions are canonical

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$\mathbb{A}$ f.b.h.: can check if map $g: \mathcal{O}(A) \to \mathcal{O}(A)$ induced by $(\mathbb{A},\mathbb{A})$-can. function $f: \mathbb{A} \to \mathbb{A}$.

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$\boxed{\text{Example}}$ $C_{\color{red}\triangle\color{blue}\triangle} = \text{Flim}\left\{\begin{array}{l}\text{2-edge-coloured graphs} \\ \text{avoiding } {\color{red}\triangle}, {\color{blue}\triangle}\end{array}\right\}$

$\mathcal{O}_1(C_{\color{red}\triangle\color{blue}\triangle}) = \{\bullet\}$ $\qquad$ $\mathcal{O}_2(C_{\color{red}\triangle\color{blue}\triangle}) = \{\bullet{=}\bullet,\ \bullet\ \bullet,\ \bullet{\color{red}-}\bullet,\ \bullet{\color{blue}-}\bullet\}$

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① $f(-) = -$, $f(-) = -$,

$f(\bullet\ \ \bullet) = \bullet\ \ \bullet$ $\quad$ valid

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Thm. (F. + Pinsker) $\mathbb{A} \leq_{f.o.} \mathbb{B}$, $\mathbb{B}$ f.b.h + Ramsey.
If $f \in \mathrm{End}(\mathbb{A})$ is $(\mathbb{B}, \mathbb{B})$-canonical and

$$f_{\mathrm{Orb}} : \mathcal{O}(\mathbb{B}) \to \mathcal{O}(\mathbb{B})$$

has inclusion-minimal range among such maps, then
$\mathbb{A}|_{\mathrm{im}(f)} = \mathbb{A}^{\mathrm{core}}$.

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"Example" $\mathbb{A} = \mathbb{B} = $ Rado. f: Rado $\to$ Rado, f(edge) = edge
f(non-edge) = edge. $\mathbb{A}|_{f(A)} = K_\omega = \text{Rado}^{core}$.

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$\mathbb{A}|_{\mathrm{im}(f)} = \mathbb{A}^{\mathrm{core}}$. $\mathbb{B}' := \mathbb{B}|_{\mathrm{im}(f)}$ is f.b.h + Ramsey
and $\mathbb{A}^{\mathrm{core}} \leq_{fo} \mathbb{B}'$. (Mottet + Pinsker 2020)

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Remark
- Can compute the bounds of $\mathbb{B}'$.
- Only fin. many $f_{\mathrm{Orb}} : \mathcal{O}(\mathbb{B}) \to \mathcal{O}(\mathbb{B})$; all computable

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Strategy:

1. Show that m-c cores are computable. ✓

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Decidability of $\simeq_{conj}$ I

Def. Let $\mathbb{A}, \mathbb{B}$ structures. Say $\mathrm{Pol}\,\mathbb{A} \simeq_{conj} \mathrm{Pol}\,\mathbb{B}$ if there is bijection $\theta : A \to B$ s.t.

$$\mathrm{Pol}\,\mathbb{A} \to \mathrm{Pol}\,\mathbb{B}, \quad f \mapsto \theta \circ f \circ (\theta^{-1}, \dots, \theta^{-1})$$

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Remark θ is such bijection iff $\forall n \geq 1$ θ induces bijection: $\{\mathrm{Pol}\,\mathbb{A}$ inv. subsets $A^n\} \to \{\mathrm{Pol}\,\mathbb{B}$ inv. subsets $B^n\}$.

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Corollary $\mathrm{Pol}\,\mathbb{A} \cong_{conj} \mathrm{Pol}\,\mathbb{B}$ iff $\exists$ signature s.t.

$$(A; \{\mathrm{Pol}\,\mathbb{A}\text{-inv. relations}\}) \cong (B; \{\mathrm{Pol}\,\mathbb{B}\text{-inv. relations}\}).$$

Decidability of $\cong_{conj}$ II

Input: $\mathbb{A} \leq_{fo} \mathbb{C}$, $\mathbb{B} \leq_{fo} \mathbb{D}$, $\mathbb{C}, \mathbb{D}$ f.b.h + Ramsey

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① Compute $\mathbb{A}^{core} \leq_{fo} \mathbb{C}'$, $\mathbb{B}^{core} \leq_{fo} \mathbb{D}'$

② Let $n \geq \max \text{arity}(\mathbb{A}^{core}, \mathbb{B}^{core})$.

- $\mathbb{A}'$ expansion of $\mathbb{A}^{core}$ by all $\mathrm{Pol}(\mathbb{A}^{core})$ inv. rel. of $\mathbb{A}^{core}$ of arity $\leq n$ (only fin. many relations)
- $\mathbb{B}'$ analogously

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Bodirsky + Pinsker + Tsankov 2017: $\mathbb{A}', \mathbb{B}'$ can be computed ($\mathbb{C}', \mathbb{D}'$ f.b.h. + Ramsey)

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③ Show that: $\operatorname{Pol} \mathbb{A}^{core} \simeq_{conj} \operatorname{Pol} \mathbb{B}^{core}$ iff $\exists$ signature for $\mathbb{A}', \mathbb{B}'$ making them isomorphic + the latter is decidable.

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THE FINAL PUSH

Thm. (Rubin 1994) If $A, \mathbb{B}$ ω-cat., without algebraicity then $\mathrm{Aut}\,\mathbb{A} \cong_{top} \mathrm{Aut}\,\mathbb{B} \Leftrightarrow \mathrm{Aut}\,\mathbb{A} \simeq_{conj} \mathrm{Aut}\,\mathbb{B}$.

"Bi-interpret. of such $\mathbb{A}, \mathbb{B}$ are bi-definitions already"

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"Bi-interpret. of such $\mathbb{A},\mathbb{B}$ are bi-definitions already"

Thm. (F.+Pinsker) If $\mathbb{A},\mathbb{B}$ ω-cat., w/o algebraicity, transitive then $\mathrm{Pol}\,\mathbb{A} \cong_{top} \mathrm{Pol}\,\mathbb{B} \Leftrightarrow \mathrm{Pol}\,\mathbb{A} \cong_{conj} \mathrm{Pol}\,\mathbb{B}$.

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Thank you!

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